

Cosmology from Group Field Theory

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(Dated: March 18, 2013)

We identify a class of condensate states in the group field theory (GFT) approach to quantum gravity that can be interpreted as macroscopic homogeneous spatial geometries. We then extract the dynamics of such condensate states directly from the fundamental quantum GFT dynamics, following the procedure used in ordinary quantum fluids. The effective dynamics is a non-linear and non-local extension of quantum cosmology. We also show that any GFT model with a kinetic term of Laplacian type gives rise, in a semi-classical (WKB) approximation and in the isotropic case, to a modified Friedmann equation. This is the first concrete, general procedure for extracting an effective cosmological dynamics directly from a fundamental theory of quantum geometry.

PACS numbers: 98.80.Qc, 04.60.Pp, 03.75.Nt

The main challenge faced by all quantum gravity approaches, beside obtaining a solid description of the fundamental degrees of freedom of quantum spacetime, is to bridge the gap between Planck-scale physics and effective physics at macroscopic scales, to provide testable predictions. In particular, in background independent approaches, it is a difficult task because the most natural notion of vacuum state is one that describes no spacetime at all, while macroscopic geometries should be thought of as states with a very large number of quantum geometric excitations [1, 2], obviously very difficult to handle in detail. Next, one needs to extract an effective dynamics for such highly excited states and relate it to the usual relativistic dynamics. This is an ever harder task, and no approach has fully succeeded, despite many interesting results in this direction [3, 4].

In this Letter, we put forward a concrete proposal to bridge this gap and, for the first time, extract an effective cosmological dynamics directly from a fundamental quantum gravity framework (as opposed to a mini-superspace reduction). We use the group field theory (GFT) formalism for quantum gravity [5], which is strictly related to loop quantum gravity and spin foam models [3], tensor models [6] and dynamical triangulations [4], including many of their mathematical ingredients. Therefore the relevance of our results extends well beyond the GFT approach.

After identifying a criterion for discrete geometries to approximate continuum ones and to be compatible with spatial homogeneity, we propose a class of GFT states describing continuum macroscopic homogeneous (but anisotropic) geometries: GFT *condensates*, superpositions of N -particle states satisfying the criterion for spatial homogeneity at each N , thus spatially homogeneous to arbitrary accuracy. The appearance of macroscopic geometries is captured by a process similar to Bose–Einstein condensation of appropriate fundamental quanta, thus realizing concretely the idea of *spacetime as*

a condensate often advocated in the past [7, 8].

We then extract the dynamics of such condensate states (in two interesting cases) directly from the fundamental quantum GFT dynamics, following the procedure used in ordinary quantum fluids. This effective dynamics is shown to have the form of a non-linear and non-local extension of quantum cosmology, similar to the one suggested in [9]. As an example, we show that any GFT model that involves a kinetic term of Laplacian type, as suggested by recent work on GFT renormalization, gives rise in a semi-classical (WKB) approximation to an Hamilton–Jacobi equation describing the classical dynamics of a homogeneous universe, and in the isotropic case to a modified Friedmann equation with corrections determined by the fundamental GFT dynamics.

Our procedure is completely general, *i.e.* it does not apply to a single GFT model, but to any model incorporating appropriate pre-geometric data, examples being [10–12]. This general procedure opens a new avenue to getting effective equations for an emergent spacetime geometry from a pregeometric scenario, and lends weight to claims that such quantum gravity models correspond to general relativity in a semiclassical continuum approximation. Full details of our calculations and results will be presented in a separate publication.

Group field theory — Group field theories are quantum field theories on group manifolds (or their Lie algebras), with a non-trivial combinatorial structure of quantum states and histories. Their quantum states are in fact 4-valent graphs labeled by group or Lie algebra elements, which can be equivalently represented as 3d cellular complexes. The quantum dynamics, in a perturbative expansion around the (‘no-space’) vacuum, gives a sum of Feynman diagrams dual to 4d cellular complexes of arbitrary topology. The Feynman amplitudes for these discrete histories can be written either as spin foam models [13] or as simplicial gravity path integrals [4]. The relation with other approaches to quantum gravity is apparent.

For technical simplicity only, we focus here on the Riemannian case. The counterpart of our construction for Lorentzian GFT models is straightforward.

In this setting GFTs can be defined in terms of a (complex) bosonic field $\varphi(g_1, g_2, g_3, g_4)$ on $\text{SO}(4)^4$, which can be expanded in annihilation operators: $\hat{\varphi}(g_I) = \sum_\nu \varphi_\nu(g_I) \hat{a}_\nu$; using the basic operators \hat{a}_ν^\dagger , one can then construct the GFT Fock space out of the ‘no-space’ vacuum $|0\rangle$. A quantum of the GFT field, created by the operator $\hat{\varphi}^\dagger(g_1, \dots, g_4)$, is interpreted as a tetrahedron whose geometry is given by the four parallel transports g_I of the gravitational $\text{SO}(4)$ connection along links dual to its faces. In this picture, a superposition of N -particle states in the GFT corresponds to a spin network with N vertices or a complex with N tetrahedra.

One can use a noncommutative Fourier transform to define the analogous field on the Lie algebra $\mathfrak{so}(4)^4$, $\tilde{\varphi}(B_1, B_2, B_3, B_4)$ [12]. The conjugate variables B_I are bivectors associated to the faces of the tetrahedron:

$$B_{\Delta_I}^{AB} \sim \int_{\Delta_I} e^A \wedge e^B. \quad (1)$$

e is a co-tetrad field encoding the simplicial geometry.

In order to ensure this interpretation, the variables B_I must satisfy two types of conditions. First, *simplicity constraints* [10–13]:

$$\exists n^A \in S^3 \subset \mathbb{R}^4 : \forall I \quad n_A B_I^{AB} = 0. \quad (2)$$

These impose a restriction on the domain of φ to a submanifold of $\text{SO}(4)$, with different constructions having been proposed [10–12]. For example [14], (2) can be imposed by requiring

$$\varphi(g_1, g_2, g_3, g_4) = \varphi(g_1 h_1, g_2 h_2, g_3 h_3, g_4 h_4) \forall h_I \in \text{SO}(3), \quad (3)$$

so that φ takes values on four copies of $\text{SO}(4)/\text{SO}(3) \sim S^3 \sim \text{SU}(2)$. For GFT models with direct relation to loop quantum gravity [10], instead, the field dependence is reduced to the diagonal $\text{SU}(2)$ subgroup of $\text{SO}(4)$.

A second condition is invariance under gauge transformations (which in the graph representation of the states, can be seen as acting on the vertex joining the dual links). This can be implemented as the invariance

$$\varphi(g_1, g_2, g_3, g_4) = \varphi(g_1 h, g_2 h, g_3 h, g_4 h) \forall h \in \text{SO}(4). \quad (4)$$

In Lie algebra variables, (4) encodes a *closure constraint*: the bivectors B_I must close to form a tetrahedron [12].

The simplicity constraints imply that there exist vectors $e_i^A \in \mathbb{R}^4$ (for $i = 1, 2, 3$) such that for all i

$$B_i^{AB} = \epsilon_i^{jk} e_j^A e_k^B. \quad (5)$$

Approximate geometries and homogeneity — In this second quantized formalism, the N -particle state

$$|B_{I(m)}\rangle := \frac{1}{N!} \prod_{m=1}^N \hat{\varphi}^\dagger(B_{1(m)}, \dots, B_{4(m)}) |0\rangle \quad (6)$$

where $|0\rangle$ is the Fock vacuum, is interpreted as a discrete geometry of N tetrahedra with bivectors $B_{I(m)}$ associated to the faces. Assuming closure and simplicity constraints, we parametrize (6) by $3N$ bivectors $\{B_{i(m)}\}$ ($i = 1, \dots, 3$, $m = 1, \dots, N$) of the form (5). On this space of bivectors, or alternatively the space of $e_{i(m)}^A$, there is an action of $\text{SO}(4)^N$,

$$B_{i(m)} \mapsto (h_{(m)})^{-1} B_{i(m)} h_{(m)}, \quad e_{i(m)} \mapsto e_{i(m)} h_{(m)}. \quad (7)$$

This is a gauge symmetry of gravity, corresponding to a local frame rotation. The gauge-invariant configuration space for each tetrahedron is six-dimensional and may be parametrized by the ‘metric’ components

$$g_{ij(m)} = e_{i(m)}^A e_{j(m)A}. \quad (8)$$

Defining the six bilinears $\tilde{B}_{ij} := B_i^{AB} B_{jAB}$, we can express the components g_{ij} in terms of the bivectors $B_{i(n)}$:

$$g_{ij} = \frac{1}{8 \text{tr}(B_1 B_2 B_3)} \epsilon_i^{kl} \epsilon_j^{mn} \tilde{B}_{km} \tilde{B}_{ln}. \quad (9)$$

We now show how to associate to these data an approximate continuum (spatial) geometry. We think of the tetrahedra as embedded into a 3-dimensional topological manifold \mathcal{M} on which a Lie group G acts transitively, so that $\mathcal{M} \simeq G/X$ where X can be a discrete subgroup of G ; G defines the notion of homogeneity. An embedding of each tetrahedron is specified by the location of one of the vertices and three tangent vectors specifying the directions of the three edges emanating from this vertex,

$$\triangle_m \mapsto \{x_m \in \mathcal{M}, \{\mathbf{v}_{1(m)}, \mathbf{v}_{2(m)}, \mathbf{v}_{3(m)}\} \subset T_{x_m} \mathcal{M}\}. \quad (10)$$

In order to exponentiate the tangent vectors to obtain the location of the other three vertices, we can use the Maurer–Cartan connection on G pulled back to \mathcal{M} .

We interpret the \mathbb{R}^4 vectors $e_{i(m)}^A$ associated to a tetrahedron as physical tetrad vectors integrated along the edges specified by $\mathbf{v}_{i(m)}$. Notice that this requires assuming that the edges are sufficiently small so that we can approximate the tetrad as constant; the tetrahedra are associated to small regions in the embedded manifold which are sufficiently flat. As a natural choice for the embedding vectors $\mathbf{v}_{i(m)}$, we use a basis of left-invariant vector fields on G , $\mathbf{v}_{i(m)} = \mathbf{e}_i(x_m)$, where $\{\mathbf{e}_i\}$ are the vector fields on \mathcal{M} obtained by push-forward of a basis of left-invariant vector fields on G . Fixing a G -invariant inner product in the Lie algebra \mathfrak{g} such a basis is unique up to the action of $\text{O}(3)$.

Within this approximation, the vectors $e_{i(m)}^A$ are related to physical tetrad vectors by $e_{i(m)}^A = e^A(x_m)(\mathbf{e}_i(x_m))$. For the $\text{SO}(4)$ invariant quantities g_{ij} , we similarly obtain

$$g_{ij(m)} = g(x_m)(\mathbf{e}_i(x_m), \mathbf{e}_j(x_m)), \quad (11)$$

thus $g_{ij(m)}$ are the metric components in the frame $\{\mathbf{e}_i\}$.

In classical relativity, a spatially homogeneous universe is characterized by a 3-dimensional Lie group G whose action on spatial hypersurfaces leaves the metric invariant, with the possible choices for G given by the Bianchi classification [15]. We can then say that a discrete geometry of N tetrahedra, specified by the data $g_{ij(m)}$, is *compatible with spatial homogeneity* if

$$g_{ij(m)} = g_{ij(k)} \quad \forall k, m = 1, \dots, N. \quad (12)$$

This criterion only uses intrinsic geometric data and does not depend on any embedding information apart from the choice of G .

Clearly, the correspondence between N -particle GFT states and continuum geometries is only approximate. It can be viewed as the information given by sampling the metric at N points.

GFT condensates as continuum homogeneous geometries — The GFT framework allows now to take two more crucial steps: 1) lift the above construction to the quantum setting; 2) take N as variable and send it to infinity. The quantum counterpart of the classical homogeneity condition becomes the requirement that the GFT N -particle state has a product structure in which *the same wave function*, depending on the data $\{B_I\}$ (or the conjugate group elements), and invariant under the transformation (7), is assigned to each GFT quantum. Then, arbitrary superpositions of such N -particle states can be considered, with N arbitrarily large. Notice that, if (12) holds for *any* N , the reconstructed spatial geometry is homogeneous to arbitrary accuracy. This is nothing else than the definition of a GFT quantum condensate state [8].

We give now two explicit examples of such GFT condensates. The simplest is a ‘single-particle’ condensate,

$$|\sigma\rangle := \exp(\hat{\sigma})|0\rangle \quad \text{with} \quad \hat{\sigma} := \int d^4g \, \sigma(g_I) \hat{\varphi}^\dagger(g_I) \quad (13)$$

if we require $\sigma(g_I k) = \sigma(g_I)$ for all $k \in \text{SO}(4)$; without loss of generality $\sigma(k' g_I) = \sigma(g_I)$ for all $k' \in \text{SO}(4)$ because of (4).

The second uses a two-particle operator:

$$|\xi\rangle := \exp(\hat{\xi})|0\rangle \quad \text{with} \quad (14)$$

$$\hat{\xi} := \frac{1}{2} \int d^4g \, d^4h \, \xi(g_I h_I^{-1}) \hat{\varphi}^\dagger(g_I) \hat{\varphi}^\dagger(h_I), \quad (15)$$

where, thanks to (4) and $[\hat{\varphi}^\dagger(g_I), \hat{\varphi}^\dagger(h_I)] = 0$, the function ξ *automatically* satisfies $\xi(g_I) = \xi(k g_I k')$ for all $k, k' \in \text{SO}(4)$. ξ is a ‘dipole’ function on the gauge-invariant configuration space of a single tetrahedron, with the same geometric data as (13), but with the advantage of having naturally the right gauge invariance, and of taking into account some limited multi-particle correlation.

Effective cosmological dynamics — We now consider a generic GFT model for 4d quantum gravity, and extract from it the effective dynamics for a homogeneous quantum space, *i.e.* for the GFT condensates defined above, following closely the standard procedures used in quantum fluids [16]. The action consists of a quadratic (kinetic) term and a higher order interaction:

$$S[\varphi] = \int d^4g \, d^4g' \, \bar{\varphi}(g_I) \mathcal{K}(g_I, g'_I) \varphi(g'_I) + \lambda \mathcal{V}[\varphi, \bar{\varphi}] \quad (16)$$

leading to the fundamental quantum equation of motion

$$\int d^4g' \, \mathcal{K}(g_I, g'_I) \hat{\varphi}(g'_I) + \lambda \frac{\delta \hat{\mathcal{V}}}{\delta \hat{\varphi}^\dagger(g_I)} = 0 \quad (17)$$

(and its complex conjugate). Now we apply these operator equations to the GFT condensates, taking into account the approximation of sufficient ‘flatness’ of the GFT building blocks required for the consistent extraction of a continuum homogeneous geometry from the condensate states.

The effective dynamics takes of course different form for the two choices of GFT states. In both cases we get an effective equation for the ‘collective cosmological wave functions’ σ or ξ . The simplest effective dynamics is obtained for the states (13). Since $|\sigma\rangle$ is an eigenstate of $\hat{\varphi}(g_I)$, when (17) acts on $|\sigma\rangle$ it becomes the non-linear equation for σ :

$$\int d^4g' \, \tilde{\mathcal{K}}(g_I, g'_I) \sigma(g'_I) + \lambda \left. \frac{\delta \tilde{\mathcal{V}}}{\delta \tilde{\varphi}(g_I)} \right|_{\varphi=\sigma} = 0, \quad (18)$$

where the $\tilde{}$ indicates that the kernels are the result of imposing the mentioned approximations to the ones appearing in the original GFT dynamics.

This is the direct GFT analogue of the Gross-Pitaevskii equation for real Bose condensates, and it is generally a non-linear and non-local (on the space of geometries B_I or g_I) equation for the collective wave function σ . It is a cosmological equation very similar to the ones of the non-linear extension of loop quantum cosmology in [9], and of the simplified GFT model [17].

The effective dynamics is extracted similarly for the state $|\xi\rangle$, inserting it in the quantum equations for GFT N -point functions. In general, the result will be a coupled set of equations involving convolutions of the cosmological wave function ξ , a non-linear and non-local extension of quantum cosmology. If the interaction \mathcal{V} is of odd order, because all odd correlation functions vanish, the two terms in (17) give independent constraints on the function ξ . The kinetic part alone gives a non-trivial condition on the wave function

$$\int d^4g'' \, \hat{\mathcal{K}}(g'_I, g''_I) \xi(g_I g''_I^{-1}) = 0. \quad (19)$$

Since (19) is linear, it could be interpreted as a standard quantum cosmological equation of motion for ξ ,

with Hamiltonian constraint given by $\tilde{\mathcal{K}}$. This equation implies that a condensation of correlated pairs of GFT quanta, for this class of GFT models (with odd interactions), is only possible if the kinetic operator $\hat{\mathcal{K}}$ admits a nontrivial kernel. The exact form of the equations of course depend on the specific GFT model considered, and for interesting models will be given in a later publication, together with the details of the above derivation.

Effective modified Friedmann equation — One can prove another interesting result, in quite some generality: any model containing a kinetic operator being the Laplace-Beltrami operator on $\text{SU}(2)^4$, together with a ‘mass term’, gives a modified Friedmann equation in the semi-classical and isotropic limit. This case is relevant because $\text{SU}(2)^4$ is a natural domain for many GFT models for 4d gravity, while the presence of the Laplacian seems to be required by GFT renormalization [18].

The effective cosmological dynamics reduces (*e.g.* in a weak-coupling limit, for the simple condensate $|\sigma\rangle$) to, or contains (for the dipole condensate $|\xi\rangle$), which we use in the following) the equation:

$$(\Delta_{g_I} + \mu) \xi(g_I g_I'^{-1}) = 0. \quad (20)$$

Using the parametrization for $\text{SU}(2)$ given by $g = \sqrt{1 - \vec{\pi}^2} \mathbf{1} - i\vec{\sigma} \cdot \vec{\pi}$, $|\vec{\pi}| \leq 1$, where σ^i are the Pauli matrices, the Laplace-Beltrami operator on $\text{SU}(2)$ is

$$\Delta_g f(\pi[g]) = (\delta^{\alpha\beta} - \pi^\alpha \pi^\beta) \partial_\alpha \partial_\beta f(\pi). \quad (21)$$

Substituting this expression into (20), rewriting $\xi(\pi_I[g_I]) = A[\pi_I] \exp(iS[\pi_I]/\kappa)$ and taking the (formal) eikonal limit $\kappa \rightarrow 0$, this equation reduces to

$$\sum_I (B_I \cdot B_I - (\pi_I \cdot B_I)^2) \approx 0, \quad (22)$$

where \cdot is the Killing form on $\mathfrak{su}(2)$ and $B_I := \partial S / \partial \pi_I$ is the momentum conjugate to π_I . Since $S[\pi(g_I)] = S[\pi(kg_I k')]$ for all k, k' in $\text{SU}(2)$ the B_I satisfy additional relations. Within this WKB approximation (22) becomes the Hamilton-Jacobi equation for the classical action S . For this scheme to be self-consistent, the phase of the function ξ has to vary rapidly compared to the modulus (which is peaked near the identity in $\text{SU}(2)^4$). (22) contains only the leading term in the WKB expansion, and the term in μ , being of higher order, does not appear.

In order to identify the B_I and π_I with cosmological variables, we write $B_I = a_I^2 T_I$, where each T_I is a dimensionless normalized Lie algebra element, $T_I \cdot T_I = 1$, and similarly $\pi_I = p_I V_I$ for normalized V_I . This identification follows from the geometric interpretation of the bivectors B_I (which encode the scale factors) and of the conjugate quantities π_I as infinitesimal holonomies. Then (22) becomes

$$\sum_I a_I^4 (p_I^2 c_I^2 - 1) \approx 0, \quad (23)$$

where $c_I = T_I \cdot V_I$ depend on the state. Specializing to an isotropic geometry, we can set $a_I = \gamma_I a$, $p_I = \beta_I p$ for constants γ_I and β_I , and (23) becomes

$$p^2 - k = O\left(\frac{\kappa}{a^2}\right), \quad (24)$$

where $k = (\sum_I \gamma_I^4) / (\sum_I \gamma_I^4 \beta_I^2 c_I^2)$. At leading order this is the classical Friedmann equation for an empty universe with spatial curvature k . Since $k > 0$, this interpretation is consistent when $G = \text{SU}(2)$. The fundamental GFT dynamics allows also to compute explicitly the corrections to such an equation, which include both the subdominant terms in the WKB approximation of the above equation, and the corrections coming from the higher order terms in the effective cosmological dynamics.

Discussion — This Letter illustrates a new and concrete avenue for extracting an effective cosmological dynamics from a fundamental (complete) quantum gravity formalism. We believe it is the first time that such a direct, simple path is open in (background independent, pre-geometric) quantum gravity approaches.

The results presented can be summarized as follows. We have identified quantum GFT states (easily exportable to the loop quantum gravity/spin foam or simplicial gravity approaches) that are natural candidates to describe homogeneous (anisotropic) cosmological geometries. They are GFT quantum condensates. Similar states have indeed been proposed in related contexts [19, 20]. Contrary to those proposals, however, the GFT condensates do not depend on any single lattice structure. The advantage of this will appear once moving away from the homogeneous condensed state: inhomogeneities in the geometry can be encoded in fluctuations above the GFT condensate states, and such coherent states support such perturbations at any approximation scale. Most importantly, condensate GFT states allowed us to extract an effective cosmological dynamics from the fundamental GFT dynamics, in full generality and rather straightforwardly. It takes the form of a non-linear and non-local extension of standard (loop) quantum cosmology, which then arises as a GFT analogue of Gross-Pitaevskii hydrodynamics in real Bose condensates. This extraction procedure can be applied to any given GFT model (with the right type of pre-geometric data), specifically to the interesting models proposed in [10, 12]. We have also shown that, for any GFT model having a kinetic term of Laplacian form, a modified Friedmann equation can be obtained in the semi-classical and isotropic limit. This new avenue should now be explored in full and points to several directions, all aimed at extracting interesting physics *directly* from current candidate GFT models for quantum gravity, thus solidly rooted in a complete quantum gravity framework, for instance, quantum gravity corrections to FRW cosmology and to the evolution of fluctuations above the GFT condensate.

At a more formal level, the ongoing work on GFT renormalization [18, 21] and phase transitions in GFT and tensor models [6, 22] can now be better directed towards proving rigorously the dynamical realization of the *condensation* leading, in the relevant GFT models, to the states (13) or (15). This will establish rigorously the physical picture of continuum space as a GFT condensate that underlies our results, and give a solid mathematical basis for the physical predictions to be obtained following the new avenue we have proposed.

Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research & Innovation. The research leading to these results has received funding from the [European Union] Seventh Framework Programme [FP7-People-2010-IRSES] under grant agreement n° 269217. SG was supported by a Riemann Fellowship of the Riemann Center for Geometry and Physics. DO acknowledges financial support from the A. von Humboldt Stiftung with a Sofja Kovalevskaja Award.

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